ADDITIONAL PROPERTIES FOR WEAKLY $P_0$
AND RELATED PROPERTIES WITH
AN APPLICATION

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Abstract

Within this paper, additional properties for weakly $P_0$ and related properties are given and the results are applied to give an internal, well-defined topological properties $WP$ for which $WP = \text{weaker } P_0$ for each $P_0$ that exists.

1. Introduction and Preliminaries

$T_0$-identification spaces were introduced by Stone in 1936 [12].

Definition 1.1. Let $(X, T)$ be a space, let $R$ be the equivalence relation on $X$ defined by $xRy$ iff $\text{Cl}([x]) = \text{Cl}([y])$, let $X_0$ be the set of $R$ equivalence classes of $X$, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on $X_0$ determined by $(X, T)$ and the map $N$. Then $(X_0, Q(X, Y))$ is the $T_0$-identification space of $(X, T)$.

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Within the 1936 paper [12], $T_0$-identification spaces were used to further characterize metrizable spaces.

**Theorem 1.1.** A space $(X, T)$ is pseudometrizable iff $(X_0, Q(X, Q(X, T)))$, is metrizable. In the 1975 paper [11], $T_0$-identification spaces were used to further characterize Hausdorff spaces.

**Theorem 1.2.** A space $(X, T)$ is weakly Hausdorff iff $(X_0, Q(X, T))$ is Hausdorff [9].

Within the 2015 paper [1], the metrizable and Hausdorff properties were generalized to weakly $P_0$ properties.

**Definition 1.2.** Let $P$ be topological properties such that $P_0 = (P$ and $T_0)$ exists. Then a space $(X, T)$ is weakly $P_0$ iff its $T_0$-identification space $(X_0, Q(X, T))$ has property $P$. A topological property $P_0$ for which weakly $P_0$ exists is called a weakly $P_0$ property [1].

In the work below, for a topological property $Q$, $(Q$ and $T_0)$ will be denoted by $Q_0$. In the 1936 paper [12], it was shown that for each space, its $T_0$-identification space has property $T_0$. Thus, for a topological property $P$ for which $P_0$ exists, a space is weakly $P_0$ iff its $T_0$-identification space has property $P_0$.

Within the paper [1], it was shown that for a weakly $P_0$ property $Q_0$, a space is weakly $Q_0$ iff its $T_0$-identification space is weakly $Q_0$, which led to the introduction and investigation of $T_0$-identification $P$ properties [2].

**Definition 1.3.** Let $S$ be a topological property. Then $S$ is a $T_0$-identification $P$ property iff both a space and its $T_0$-identification space simultaneously shares property $S$. 
In the introductory weakly $Po$ paper [1], it was shown that weakly $Po$ is neither $T_0$ nor “not-$T_0$”, where “not-$T_0$” is the negation of $T_0$. The need and use of “not-$T_0$” revealed “not-$T_0$” as a useful topological property and tool, motivating the inclusion of the long-neglected properties “not-$P$”, where $P$ is a topological property for which “not-$P$” exists, as important properties for investigation and use in the study of topology. As a result, within a short time period, many new, important, fundamental, foundational, never before imagined properties have been discovered, expanding and changing the study of topology.

In past studies of weakly $Po$ spaces and properties, for a classical topological property $Qo$, a special topological property $W$ was sought such that for a space with property $W$, its $T_0$-identification space has property $Qo$, which then implies the initial space has property $W$. If past practices continue, the study of weakly $Po$ spaces and properties would continue to be tedious and never ending. Thus, the question of whether there is a shortcut for the weakly $Po$ space and property search process arose, which was resolved in a recent paper [3].

**Answer 1.1.** Let $Q$ be a topological property for which both $Qo$ and $(Q \text{ and } \text{“not-$T_0$”})$ exist. Then $Q$ is a $T_0$-identification $P$ property that is weakly $Po$ and $Q = \text{weakly } Qo = (Qo \text{ or } (Q \text{ and } \text{“not-$T_0$”}))$ [3].

**Answer 1.2.** $\{Q \mid Q \text{ is a } T_0\text{-identification } P \text{ property}\} = \{Qo \mid Qo \text{ is a weakly } Po \text{ property}\} = \{Qo \mid Q \text{ is a topological property and } Qo \text{ exists}\} [3].$

**Answer 1.3.** $\{Q \mid Q \text{ is a } T_0\text{-identification } P \text{ property }\} = \{Q \mid Q \text{ is weakly } Po\} = \{Q \mid Q \text{ is a topological property and both } Qo \text{ and } (Q \text{ and } \text{“not-$T_0$”}) \text{ exist}\} [3].$

Thus, major progress was achieved in the study of weakly $Po$ and related properties. If $Q$ is a topological property for which both $Qo$ and
If $Q$ is a topological property for which $Q_0$ exists, then $Q = Q_0$ is a weakly Po property, but $Q = Q_0$ is not a $T_0$-identification $P$ property or weakly Po. Within the recent paper [3], a topological property $W$ that can be both $T_0$ and “not-$T_0$” was given that is a $T_0$-identification $P$ property that is weakly Po such that $W = \text{weakly } Q_0$, again making the search process certain, and quick and easy.

Definition 1.4. Let $Q$ be a topological property for which $Q_0$ exists. A space $(X, T)$ has property $QNO$ iff $(X, T)$ is “not-$T_0$” and $(X_0, Q(X, T))$ has property $Q_0$ [3].

In the recent paper [3], it was shown that $QNO$ exists and is a topological property, and $W = (Q_0 \text{ or } QNO)$ is a $T_0$-identification $P$ property that is weakly Po with $W = \text{weakly } Q_0$. Thus, as stated above, for a topological property $Q$ for which $Q_0$ exists and $Q = Q_0$, there is a certain, quick, and easy answer.

In the weakly Po paper [4], the use of $T_0$ and “not-$T_0$” revealed that $L = (T_0 \text{ or } \text{not-$T_0$})$ is the least of all topological properties. Since the existence of the least topological property $L$ had not even been considered in past studies, its existence required a change in the definition of subspace properties with the removal of $L$ as a subspace property [5]. Prior to the study of weakly Po spaces and properties, it was unknown whether for subspace properties $P$ and $Q$, if $(P \text{ and } Q)$ exists. Within the 2016 paper [5], it was shown that for subspace properties $P$ and $Q$, $(P \text{ and } Q)$ exists and, thus, is a subspace property. In the paper [5], it was shown that $\mathcal{P} = \{P \mid P \text{ is a subspace property}\}$ has no least element and in the paper [6], it was shown that $\mathcal{P}$ has strongest element the singleton set property. Since, for the most part, “not-$P$”, where $P$ is a topological
property for which “not-$P$” exists, had been ignored in the study of topology, the introduction and investigation of “not-(subspace property $P$)” in the paper [4] revealed that for a subspace property $P$, “not-(subspace property $P$)” is not a subspace property, which instantaneously gave many previously unknown examples of topological properties that are not subspace properties. Also, in the paper [4], it was shown that for subspace properties $P$ and $Q$ for which $(P$ and “not-$Q$”) exist, $(P$ and “not-$Q$”) is not a subspace property, giving many additional new topological properties that are now known to be not subspace properties. Within the 2017 paper [7], similar results were obtained for product properties.

Thus, the study of weakly $P_0$ and related properties has exposed a previously unknown, very fruitful territory within topology with properties and tools needed to successfully address previously unasked and/or unanswered questions. Below, the exploration of the recently discovered new territory within topology continues.

2. Additional Properties of Weakly $P_0$
and Related Properties

Within the paper [8], it was shown that $L$ is weakly $P_0$ and $L_0 = T_0$ is a weakly $P_0$ property. In the paper [9], it was shown that for a “not-$T_0$” space $(X, T)$, there exists a proper subspace $(X_{TO}, T_{XTO})$ that is homeomorphic to $(X_0, Q(X, T))$.

**Definition 2.1.** Let $(X, T)$ be a “not-$T_0$” space and let $C_x$ be the $T_0$-identification space equivalence class containing $x$. Then $X_{TO}$ is a subset of $X$ that contains exactly one element from each equivalence class $C_x$ [9].
**Theorem 2.1.** Let \((X, T)\) be “not-T\(_0\)” and let \(X TO\) be as defined above. Then \((X TO, T_{X TO})\) is homeomorphic to \((X_0, Q(X, T))\) [9].

Below, the results above are used to further investigate weakly Po and related properties.

**Theorem 2.2.** Let \((X, T)\) be a “not-T\(_0\)” space and let \(X TO\) be as defined above. Then \((X TO, T_{X TO})\) is \(T_0\).

**Proof.** Since \((X_0, Q(X, T))\) is \(T_0\) and \(T_0\) is a topological property, then, by the results above, \((X TO, T_{X TO})\) is \(T_0\).

**Theorem 2.3.** For each space \((X, T)\) that is “not-T\(_0\)” \((X TO, T_{X TO})\) is a proper, dense subspace of \((X, T)\) that is \(T_0\).

**Proof.** By the results above, \(X TO\) is a proper subset of \(X\) and \((X TO, T_{X TO})\) is \(T_0\). Let \(x \in X \setminus X TO\). Let \(O \in T\) such that \(x \in O\). Let \(y \in X TO\) such that \(y \in C_x\). Since \(Cl(\{x\}) = Cl(\{y\})\), then \(y \in O\). Hence, \(X TO\) is dense in \((X, T)\).

**Theorem 2.4.** Let \(Q\) be a topological property for which \((Q \text{ and “not-T\(_0\)”})\) exists. Then the following are equivalent: (a) \(Q\) is a \(T_0\)-identification \(P\) property, (b) \(Q\) is a weakly Po property, (c) for each space \((X, T)\) with property \((Q \text{ and “not-T\(_0\)”})\), \((X TO, T_{X TO})\) has property \(Q\), and (d) for each space \((X, T)\) with property \((Q \text{ and “not-T\(_0\)”})\), \((X TO, T_{X TO})\) has property \(Q_0\).

**Proof.** By the results above, (a) and (b) are equivalent.

(b) implies (c): Let \((X, T)\) be a space with property \((Q \text{ and “not-T\(_0\)”})\). Since \(Q\) is weakly \(Q_0\), then \((X_0, Q(X, T))\) has property \(Q\). Since \((X, T)\) has property “not-T\(_0\)” then \((X TO, T_{X TO})\) is...
homeomorphic to \((X_0, Q(X, T))\) and since \(Q\) is a topological property, 
\((X_{TO}, T_{XTO})\) has property \(Q\).

By Theorem 2.2, (c) implies (d).

(d) implies (b): Let \((X, T)\) be a space with property \((Q \text{ and not-}T_0")\).
Since \((X_{TO}, T_{XTO})\) has property \(Q_0\), then both \(Q_0\) and \((Q \text{ and not-}T_0")\)
exist and, by Answer 1.3 above, \(Q\) is weakly \(P_0\).

In the 2015 paper [2], it was proven that for topological properties \(Q\) and \(W\), which are both weakly \(P_0\), weakly \(Q_0 = \text{weakly } W_0\) iff \(Q_0 = W_0\).
Could \(Q = W\) be added as an equivalent statement?

**Theorem 2.5.** Let \(Q\) be a topological property that is weakly \(P_0\). Then \(Q\) is a \(T_0\)-identification \(P\) property and \(Q = \text{weakly } Q_0\).

**Proof.** By Answer 1.3 above, both \(Q_0\) and \((Q \text{ and not-}T_0")\) exist. 
Then by Answer 1.1 above, \(Q\) is a \(T_0\)-identification \(P\) property and \(Q = \text{weakly } Q_0\).

**Theorem 2.6.** Let \(Q\) and \(W\) be topological properties, both of which are weakly \(P_0\). Then the following are equivalent: (a) weakly \(Q_0 = \text{weakly } W_0\), (b) \(Q_0 = W_0\), and (c) \(Q = W\).

**Proof.** By the results above, (a) and (b) are equivalent.

(a) implies (c): By Theorem 2.5, \(Q = \text{weakly } Q_0\) and \(W = \text{weakly } W_0\) 
and, since weakly \(Q_0 = \text{weakly } W_0\), then \(Q = W\).

Clearly, (c) implies (a).

**Theorem 2.7.** Let \(Q\) and \(W\) be topological properties, both of which are weakly \(P_0\). Then the following are equivalent: (a) weakly \(Q_0\) implies weakly \(W_0\), (b) \(Q\) implies \(W\), and (c) \(Q_0\) implies \(W_0\).

**Proof.** (a) implies (b): Since \(Q = \text{weakly } Q_0\), \(W = \text{weakly } W_0\), and weakly \(Q_0\) implies weakly \(W_0\), then \(Q\) implies \(W\).
Clearly (b) implies (c).

(c) implies (a): Let \((X, T)\) be weakly \(Q_0\). Then \((X_0, Q(X, T))\) is \(Q_0\), which implies \((X_0, Q(X, T))\) is \(W_0\), which implies \((X, T)\) is weakly \(W_0\). Thus, weakly \(Q_0\) implies weakly \(W_0\).

3. Weakly \(P_0\) for Each Existent \(Q_0\)

**Definition 3.1.** A space \((X, T)\) has property \(SM\) iff for \(x\) and \(y\) in \(X\) such that \(Cl(\{x\}) \neq Cl(\{y\})\), there exists an open set \(U\) containing only one of \(x\) and \(y\).

**Theorem 3.1.** Every space has property \(SM\).

**Proof.** Let \((X, T)\) be a space. Let \(x\) and \(y\) be elements in \(X\) such that \(Cl(\{x\}) \neq Cl(\{y\})\). Then \(x \notin Cl(\{y\})\) or \(y \notin Cl(\{x\})\), say \(x \notin Cl(\{y\})\). Then \(x \in U = X \setminus Cl(\{y\})\) is open and \(y \notin U\). Thus \((X, T)\) has property \(SM\).

**Corollary 3.1.** \(SM\) is a topological property.

**Theorem 3.2.** Let \((X, T)\) be a space. Then \((X, T)\) has property \(L\) iff for \(x\) and \(y\) in \(X\) such that \(Cl(\{x\}) \neq Cl(\{y\})\), there exists an open set \(U\) containing only one of \(x\) and \(y\).

**Proof.** By Theorem 3.1, if \((X, T)\) has property \(L\), then \((X, T)\) has property \(SM\). Thus \(L\) implies \(SM\) and, since \(L\) is the least of all topological properties, then \(L = SM\).

**Definition 3.2.** Let \((X, T)\) be a space and for each \(x \in X\), let \(C_x\) be the \(T_0\)-identification class containing \(x\). Then \(O_X T_0\) is a subset of \(X\) containing exactly one element from each equivalence class \(C_x\).

Note that if \((X, T)\) is “not-\(T_0\)” then a subset \(O_X T_0\) of \(X\) is an \(X_T\) subset of \(X\).
Theorem 3.3. Let $(X, T)$ be a space. Then $(OXTO, TOXTO)$ is homeomorphic to $(X_0, Q(X, T))$.

Proof. By Theorem 3.2, $(X, T)$ is $(T_0$ or “not-$T_0$”). Consider the case that $(X, T)$ is $T_0$. Then $X = OXTO$ and $(X, T) = (OXTO, TOXTO)$ is $T_0$. Since a space $(Y, S)$ is $T_0$ iff the natural map $N : (Y, S) \to (Y_0, Q(Y, S))$ is a homeomorphism [10], then $(OXTO, TOXTO)$ is homeomorphic to $(X_0, Q(X, T))$. Thus, consider the case that $(X, T)$ is “not-$T_0$”. Then $OXTO = XTO$ and, by Theorem 2.1, $(OXTO, TOXTO)$ is homeomorphic to $(X_0, Q(X, T))$.

Definition 3.3. Let $Q$ be a topological property for which $Q_0$ exists. Then a space $(X, T)$ has property $WQ$ iff $(OXTO, TOXTO)$ has property $Q_0$.

Within the paper [3], it was shown that for a topological property $Q$ such that $Q_0$ exists, $(Q_0$ or $QNO)$ is a $T_0$-identification $P$ property, $(Q_0$ or $QNO) = \text{weakly} (Q_0$ or $QNO)_0 = \text{weakly} Q_0$, and $(Q_0$ or $QNO)$ is a weakly $Po$ property, which is used below.

Theorem 3.4. Let $Q$ be a topological property for which $Q_0$ exists and let $(X, T)$ be a space. Then $(X, T)$ has property $WQ$ iff $(X, T)$ has property $(Q_0$ or $QNO)$.

Proof. Suppose $(X, T)$ has property $WQ$. Then $(OXTO, TOXTO)$ has property $Q_0$ and, since $(OXTO, TOXTO)$ is homeomorphic to $(X_0, Q(X, T))$, then $(X_0, Q(X, T))$ has property $Q_0$. Since $Q_0$ exists, then $(Q_0$ or $QNO)$ is a $T_0$-identification $P$ property and $(Q_0$ or $QNO) = \text{weakly} Q_0$. Thus, a space $(Y, S)$ is $(Q_0$ or $QNO)$ iff $(Y_0, Q(Y, S))$ is $Q_0$, and, since $(X_0, Q(X, T))$ is $Q_0$, then $(X, T)$ is $(Q_0$ or $QNO)$.
Conversely, suppose \((X, T)\) has property \((Q_0 \text{ or } Q_{NO})\). Then \((X_0, Q(X, T))\) has property \(Q_0\) and, since \((OXTO, T_{OXTO})\) is homeomorphic to \((X_0, Q(X, T))\), then \((OXTO, T_{OXTO})\) has property \(Q_0\). Hence \((X, T)\) has property \(W_Q\).

Therefore, \(W_Q = (Q_0 \text{ or } Q_{NO})\).

Since both \(Q_0\) and \(Q_{NO}\) are topological properties, then \(W_Q\) is a topological property.

**Corollary 3.2.** Let \(Q\) be a topological property for which \(Q_0\) exists. Then \(W_Q\) is the special property for which a space has property \(W_Q\) iff its \(T_0\)-identification space has property \(Q_0\).

Below the results above are applied to determine \(W(T_1)\).

Let \((X, T)\) have property \(W(T_1)\).

Then \((OXTO, T_{OXTO})\) has property \((T_1)\) or \(T_1\). Suppose there exists a \(x \in X\) such that \(C_x \neq Cl(\{x\})\). Since \(C_x \subseteq Cl(\{x\})\), let \(y \in Cl(\{x\})\) that is not in \(C_x\). Let \(u, v \in OXTO\) such that \(x \in C_u\) and \(y \in C_v\). Then \(u\) and \(v\) are distinct elements in \(OXTO\) and since \((OXTO, T_{OXTO})\) is \(T_1\), there exists and open set \(V\) containing \(v\) and not \(u\). Let \(O \in T\) such that \(V = O \cap OXTO\). Then \(y \in O\) and \(x \notin O\), which contradicts \(y \in Cl(\{x\})\).

Thus, for each \(x \in X\), \(C_x = Cl(\{x\})\) and \(\{Cl(\{x\})| x \in X\}\) is a decomposition of \(X\).

Conversely, suppose \(\{Cl(\{x\})| x \in X\}\) is a decomposition of \(X\). Let \(u\) and \(v\) be distinct elements of \(OXTO\). Then \(\{v\} = Cl(\{v\}) \cap OXTO\) is closed in \(OXTO\). Hence singleton sets are closed in \(OXTO\) and \((OXTO, T_{OXTO})\) is \(T_1\).
Hence, a space \((X, T)\) has property \(W(T_1)\) iff \(\{\text{Cl}(\{x\})| x \in X\}\) is a decomposition of \(X\).

Past work in \(T_0\)-identification spaces and weakly \(Po\) spaces verify the result above. The application above is not required, but it does add comfort to use of the work above.

In the paper [3], it was shown that if \(Q\) is a topological property for which both \(Qo\) and \((Q \text{ and } \text{not-}T_0\)”) exist, then \(Q\) is a \(T_0\)-identification \(P\) property, \(QNO = (Q \text{ or } \text{”not-}T_0\)”), and \(Q = \text{weakly } Qo = (Qo \text{ or } QNO) = (Qo \text{ or } (Q \text{ and } \text{”not-}T_0\)”}), which greatly simplifies the above process.

**Corollary 3.3.** Let \(Q\) be a topological property for which both \(Qo\) and \((Q \text{ and } \text{”not-}T_0\)”) exist. Then \(WQ = Q\).

**References**


