ON CLOSEDNESS OF THE RANGE OF THE OPERATOR $X \mapsto TX - XT$ DEFINED ON $C_2(H)$ WHEN $T$ IS $M$-HYPONORMAL

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Abstract

We show how a proof of Stampfli can be extended to prove that the operator $X \mapsto TX - XT$ defined on the Hilbert-Schmidt class, when $T$ is an $M$-hyponormal operator, has a closed range, if and only if $\sigma(T)$ is finite.

1. Introduction

Let $H$ be a complex, separable, infinite dimensional Hilbert space, let $\mathcal{L}(H)$ denote the algebra of all linear bounded operators on $H$. The Hilbert-Schmidt class, denoted by $C_2(H)$, is a Hilbert space with the $\| \cdot \|_2$ norm that arises from the inner product $\langle X, Y \rangle = \text{tr}(XY^*)$, where $\text{tr}$ is the scalar-valued trace. For $T \in \mathcal{L}(H)$, define $\Delta_T : \mathcal{L}(H) \to \mathcal{L}(H)$ by $\Delta_T(X) = TX - XT$, and let $\sigma(T)$ denote the spectrum of $T$. Let the range of a linear operator $S$ be denoted by $\mathcal{R}(S)$. For a normal operator $N \in \mathcal{L}(H)$, Anderson and Foiaş [1] proved that $\mathcal{R}(\Delta_N)$ is norm closed, if and

2010 Mathematics Subject Classification: Primary 47B20.
Keywords and phrases: $M$-hyponormal operators, Hilbert-Schmidt class, closed range.
Received April 15, 2009

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only if $\sigma(N)$ is a finite set. In [3], Stampfli extended this result to the class of hyponormal operators.

**Theorem A** ([3]). Let $T \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator. Then $\mathcal{R}(\Delta_T)$ is norm closed, if and only if $\sigma(T)$ is finite.

In fact, Stampfli provided a proof of the “only if” implication which can be extended to a larger class of operators than hyponormal operators. For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_{nap}(T)$ denote its normal approximate point spectrum, that is, the set of those $\lambda \in \mathbb{C}$ for which there exists an orthonormal sequence $\{\phi_n\}$ in $\mathcal{H}$ such that

$$\|T - \lambda\phi_n\| + \|(T - \lambda)^*\phi_n\| \to 0.$$  

Define the class $\mathcal{G}(\mathcal{H})$ as follows:

$$\mathcal{G}(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) | \sigma_{nap}(T) \text{ is an infinite set}\}.$$  

Some classes of hyponormal related operators, such as $M$-hyponormal operators, i.e.,

$$m \cdot \|(T - \lambda)^*\phi\| \leq \|T - \lambda\phi\|, \quad \forall \phi \in \mathcal{H}, \text{ and } \forall \lambda \in \mathbb{C}, \text{ for some } m > 0,$$

have spectrum that is finite or they belong to $\mathcal{G}(\mathcal{H})$. In particular, the hyponormal operators (that is, 1-hyponormal) have this property.

In [2], Stampfli proved the following lemma which will be used in Section 2.

**Lemma B.** Let $T \in \mathcal{G}(\mathcal{H})$ and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of distinct points of $\sigma_{nap}(T)$. Then for any sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers converging to zero, there exists an orthonormal sequence $\{\phi_n\}_{n=1}^\infty$ of vectors in $\mathcal{H}$ such that

$$\|T - \lambda_n\phi_n\| + \|(T - \lambda_n)^*\phi_n\| < \varepsilon_n, \quad \text{for } n = 1, 2, \ldots, \text{ and}  \tag{1}$$

$$\langle \phi_n, T\phi_k \rangle = 0, \quad \text{for } k = 1, \ldots, n - 1. \tag{2}$$
2. The Closedness of the Range of $\Delta_T^{(2)}$

The operator $\Delta_T$ defined on the Hilbert-Schmidt class will be denoted in the remainder of this note by $\Delta_T^{(2)}$, that is, $\Delta_T^{(2)} : C_2(\mathcal{H}) \to C_2(\mathcal{H})$, $\Delta_T^{(2)}(X) = TX - XT$. Let $H^M(\mathcal{H})$ denote the set of $M$-hyponormal operators.

**Proposition 1.** Let $T \in H^M(\mathcal{H})$. If $\sigma(T)$ is finite, then $\mathcal{R}(\Delta_T^{(2)})$ is closed.

**Proof.** It is well known that an operator $T \in H^M(\mathcal{H})$ with finite spectrum is normal. Indeed, for the such an operator, the restriction to an invariant subspace $\mathcal{M}$ belongs to $H^M(\mathcal{M})$. On the other hand, if $T \in H^M(\mathcal{H})$ with $\sigma(T) = \{\lambda\}$, then $T = \lambda I$, (cf. [4]). Thus, we can write $T = \sum_{i=1}^{n_0} \lambda_i E_i$, where $E_i$’s are the spectral projections.

Let $X_n$ and $C$ be in $C_2(\mathcal{H})$ such that $\|\Delta_T^{(2)}(X_n) - C\|_2 \to 0$. Therefore, $\Delta_T(X_n) - C \to 0$ in the $\mathcal{L}(\mathcal{H})$ norm, and according to Theorem A, there exists $X^0 \in \mathcal{L}(\mathcal{H})$ such that $C = TX^0 - X^0T$. For an arbitrary $X \in \mathcal{L}(\mathcal{H})$, let $[X_{ij}]$ be the block-matrix representation of $X$ relative to the decomposition $\mathcal{H} = \sum_{i=1}^{n_0} \oplus E_i \mathcal{H}$. Thus

$$C_{ij} = (\lambda_i - \lambda_j) X^0_{ij},$$

for all $i, j = 1, \ldots, n_0$. This implies that each $X^0_{ij} = \frac{1}{(\lambda_i - \lambda_j)} C_{ij}$ is a Hilbert-Schmidt operator. Moreover, $X^0_{ii}$ can be chosen 0, and thus $X^0 \in C_2(\mathcal{H})$.

**Proposition 2.** Let $T \in \mathcal{G}(\mathcal{H})$. Then $\mathcal{R}(\Delta_T^{(2)})$ is not closed.

**Proof.** We will use same notation and circle of ideas as in [2]. Let $\{\lambda_n\}_{n \geq 1}$ be sequence of distinct points of $\sigma_{nap}(T)$ so that $\lambda_n \to \lambda_0$. Let
\[ \eta_n = \max \{|\lambda_{j+1} - \lambda_j|^{\frac{1}{2}}| j = 1, \ldots, n\}, \]

and choose a non-increasing sequence \( \{\varepsilon_n\}_{n \geq 1} \) so that \( 0 < \varepsilon_n \leq |\lambda_{n+1} - \lambda_n|^2, \) \( n \geq 1, \) and \( \sum_{n \geq 1} \varepsilon_n^2 \eta_n^2 < \infty. \) According to Lemma B, there exists an orthonormal sequence \( \{\phi_n\}_{n \geq 1} \) that satisfies (1), (2). Let \( \mathcal{H}_1 = \vee\{\phi_n| n \geq 1\}, \mathcal{H}_2 = \mathcal{H}_{-1}, \) and let \( \delta_n \) such that

\[ T\phi_n = \mu_n \phi_n + \delta_n \text{ and } \delta_n \perp \phi_n, \ n \geq 1. \] (3)

It results that

\[ |\mu_n - \lambda_n| < \varepsilon_n \text{ and } \|\delta_n\| < 2\varepsilon_n, \ n \geq 1. \] (4)

Define \( V : \mathcal{H} \to \mathcal{H} \) by \( V\phi_n = |\lambda_{j+1} - \lambda_j|^{\frac{1}{2}} \phi_{n+1}, \ n \geq 1, \) and \( Vg = 0, g \in \mathcal{H}_2. \) Let \( \mathcal{M}_n = \vee\{\phi_j| j = 1, \ldots, n\} \) and let \( P_n \) be the orthogonal projection onto \( \mathcal{M}_n \) and define \( V_n = V P_n. \) A tedious calculation shows that

\[ \Delta_T(V_n)\phi_j = \begin{cases} v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} - V_n\delta_j, & j \leq n, \\ -V_n\delta_j, & j > n, \end{cases} \]

where \( v_j = |\lambda_{j+1} - \lambda_j|^{\frac{1}{2}}. \) Denoting \( \Delta_T(V_n) - \Delta_T(V_m) \) by \( \Delta_T^{n,m}, \) then for \( n < m, \)

\[ \Delta_T^{n,m}\phi_j = \begin{cases} 0, & j \leq n, \\ -v_j(\mu_{j+1} - \mu_j)\phi_{j+1} \\ +v_j\delta_{j+1} + (V_m - V_n)\delta_j, & n < j \leq m, \\ (V_m - V_n)\delta_j, & j > m. \end{cases} \] (5)

Furthermore, from (3), it results that

\[ \delta_j \perp \phi_j, \phi_{j+1}, \phi_{j+2}, \ldots \] (6)

and from (4),
\[ \| V_n \delta_j \| \leq 2 \eta_j \varepsilon_j, \quad \text{for all } j, n \geq 1. \quad (7) \]

We will show next that \( \| \Delta_T^{n,m} \|_2 \to 0 \), when \( m, n \to \infty \). Thus, there exists \( C \in C_2(\mathcal{H}) \) such that \( \| \Delta_T(V_n) - C \|_2 \to 0 \), that is, \( C \in \mathcal{R}(\Delta_T^{(2)}) \).

First, we will show that \( \| \Delta_T^{n,m} \|_{\mathcal{H}_1}^2 \to 0 \), when \( m, n \to \infty \). Indeed,

\[
\| \Delta_T^{n,m} \|_{\mathcal{H}_1}^2 = \sum_{j=1}^{\infty} \| \Delta_T^{n,m} \phi_j \|^2 = \\
= \sum_{j=n+1}^{m} \| -v_j (\mu_{j+1} - \mu_j) \phi_{j+1} + v_j \delta_{j+1} + (V_m - V_n) \delta_j \|^2 \\
+ \sum_{j=m+1}^{\infty} \| (V_m - V_n) \delta_j \|^2.
\]

The first sum of the right hand side of the above can be majorized by

\[
2 \cdot \sum_{j=n+1}^{m} \| -v_j (\mu_{j+1} - \mu_j) \phi_{j+1} + v_j \delta_{j+1} \|^2 + 2 \cdot \sum_{j=n+1}^{m} \| (V_m - V_n) \delta_j \|^2.
\]

Since \( \phi_{j+1} \perp \delta_{j+1} \), we have

\[
\| \Delta_T^{n,m} \|_{\mathcal{H}_1}^2 \leq 2 \sum_{j=n+1}^{m} (v_j^2 |\mu_{j+1} - \mu_j|^2 + v_j^2 \| \delta_{j+1} \|^2) + \sum_{j=n+1}^{\infty} \| (V_m - V_n) \delta_j \|^2.
\]

According to (7),

\[
\| (V_m - V_n) \delta_j \|^2 \leq 16 \eta_j^2 \varepsilon_j^2,
\]

and according to (4),

\[
v_j^2 \| \delta_{j+1} \|^2 \leq 4 \eta_j^2 \varepsilon_{j+1}^2 \leq 4 \eta_j^2 \varepsilon_j^2,
\]

and

\[
|\mu_{j+1} - \mu_j|^2 \leq (2 \varepsilon_j + |\lambda_{j+1} - \lambda_{j}|)^2 \leq 8 \varepsilon_j^2 + 2 |\lambda_{j+1} - \lambda_{j}|^2.
\]
which implies

\[ v_j^2 |\mu_{j+1} - \mu_j|^2 \leq 8\eta_j^2 e_j^2 + 2|\lambda_{j+1} - \lambda_j|. \]

Therefore

\[ \|\Delta^n_{T^{-1}}/\|_{\mathcal{H}_2}^2 \leq c_1 \cdot \sum_{j=n+1}^{\infty} \eta_j^2 e_j^2 + c_2 \cdot \sum_{j=n+1}^{m} |\lambda_{j+1} - \lambda_j|, \]

where \( c_1 \) and \( c_2 \) are some constants. After a careful review of the proof, one can see that the sequence \( \{\lambda_n\} \) can be assumed to converge fast enough (otherwise, choose a subsequence of it), more precisely

\[ \sum_{j=n+1}^{m} |\lambda_{j+1} - \lambda_j| \to 0, \text{ when } n, m \to \infty. \]

We show next that \( \|\Delta^n_{T^{-1}}/\|_{\mathcal{H}_2}^2 \to 0, \) when \( m, n \to \infty. \) Indeed, we can write

\[ T^* \phi_n = \bar{\mu}_n \phi_n + \gamma_n \quad \text{with} \quad \langle \gamma_n, \phi_n \rangle = 0, \quad \text{and} \quad \|\gamma_n\| \leq 2e_n, \quad n \geq 1. \]  

\[ \text{(8)} \]

Obviously, we can write \( T^* \phi_n = \theta_n \phi_n + \gamma_n \) with \( \langle \gamma_n, \phi_n \rangle = 0, \) which implies

\[ \theta_n = \langle \theta_n \phi_n + \gamma_n, \phi_n \rangle = \langle T^* \phi_n, \phi_n \rangle = \langle \phi_n, T\phi_n \rangle = \langle \phi_n, \mu_n \phi_n + \delta_n \rangle = \bar{\mu}_n, \]

and \( \|\gamma_n\| = \| (T^* - \bar{\mu}_n) \phi_n \| \leq \| (T - \bar{\lambda})^* \phi_n \| + |\bar{\lambda} - \bar{\mu}_n| \leq 2e_n. \)

For an orthonormal basis \( \{\psi_i\}_{i \geq 1} \) of \( \mathcal{H}_2, \) we will show that

\[ \sum_{i=1}^{\infty} \|\Delta^n_{T^{-1}}/\psi_i\|^2 \to 0, \text{ when } n, m \to \infty. \]

For each \( i, \) write \( T\psi_i = \sum_{k=1}^{\infty} a_k^{(i)} \phi_k + w_i \) with \( w_i \in \mathcal{H}_2. \) Thus

\[ V_m T\psi_i = \sum_{k=1}^{m} a_k^{(i)} V_m \phi_k + V_m w_i = \sum_{k=1}^{m} a_k^{(i)} \psi_k \phi_{k+1}. \]
Since \( V_m \psi_i = 0 \), we have \( \Delta_T(V_m) \psi_i = -V_m T \psi_i \), and consequently, for \( n < m \),

\[
\Delta_T^{n, m} \psi_i = \sum_{k=n+1}^{m} a_k^{(i)} v_k \phi_{k+1}.
\]

Since the sequence \( \{\phi_k\} \) is orthonormal, we have

\[
\|\Delta_T^{n, m} \psi_i\|^2 = \sum_{k=n+1}^{m} |a_k^{(i)}|^2 \cdot v_k^2.
\]

Therefore

\[
\sum_{i=1}^{\infty} \|\Delta_T^{n, m} \psi_i\|^2 = \sum_{i=1}^{\infty} \sum_{k=n+1}^{m} |a_k^{(i)}|^2 \cdot v_k^2 = \sum_{k=n+1}^{m} v_k^2 (\sum_{i=1}^{\infty} |a_k^{(i)}|^2).
\]

For a fixed \( k \),

\[
\sum_{i=1}^{\infty} |a_k^{(i)}|^2 = \sum_{i=1}^{\infty} \langle (T \psi_i, \phi_k) \rangle^2 = \sum_{i=1}^{\infty} \langle (\psi_i, T^* \phi_k) \rangle^2 = \sum_{i=1}^{\infty} \langle \psi_i, \gamma_k \rangle^2 = \gamma_k^2 \leq 4v_k^2.
\]

Consequently, \( \sum_{i=1}^{\infty} \|\Delta_T^{n, m} \psi_i\|^2 \leq 4 \sum_{k=n+1}^{m} v_k^2 \cdot v_k^2 \to 0 \), for \( n, m \to \infty \).

The operator \( C \) is not in \( R(\Delta_T^{(2)}) \) since, according to the proof of Theorem A in [3], \( C \notin R(\Delta_T) \).

**Theorem 3.** Let \( T \in H^M(\mathcal{H}) \). Then \( R(\Delta_T^{(2)}) \) is closed, if and only if \( \sigma(T) \) is finite.

**Proof.** If \( T \in H^M(\mathcal{H}) \) and \( \sigma(T) \) is finite, then according to Proposition 1, \( R(\Delta_T^{(2)}) \) is closed. Conversely, if \( T \in H^M(\mathcal{H}) \) has an infinite spectrum, then there are infinitely many distinct points \( \{\lambda_n\} \) that are either isolated points of the spectrum, in which case they are
eigenvalues, or accumulation points of the spectrum, in which case they are in the $\sigma_{ap}(T)$. Since $T \in H^M(\mathcal{H})$, we have $\sigma_p(T), \sigma_{ap}(T) \subseteq \alpha_{nap}(T)$. Thus $T \in \mathcal{G}(\mathcal{H})$, and according to Proposition 2, $\mathcal{R}(\Delta_{T}^{(2)})$ is not closed.

References


