FINITE GROUPS WITH SOME SUBGROUPS OF SYLOW SUBGROUPS S-SUPPLEMENTED

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Abstract

A subgroup $H$ is called $s$-supplemented in $G$, if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_{SG}$, where $H_{SG}$ is the largest subnormal subgroup of $G$ contained in $H$. In this paper, we investigate the influence of $s$-supplemented primary subgroups in finite groups. Some new results about $p$-nilpotency of finite groups are obtained.

1. Introduction

Let $H$ be a subgroup of $G$. Then, a subgroup $K$ of $G$ is called a supplement of $H$ in $G$, if $G = HK$. It is of interest to study the structure
of the group by using supplement of subgroups of finite groups. In [7] and [8], Kegel proved that a group \( G \) is soluble, if every maximal subgroup of \( G \) either has a cyclic supplement in \( G \) or if some nilpotent subgroup of \( G \) has a nilpotent supplement in \( G \). On the other hand, by the well-known Hall's theorem [6], a group \( G \) is soluble, if and only if every Sylow subgroup of \( G \) has a complement in \( G \). Recently, in [12, 13], Wang obtained some new characterizations for soluble and supersoluble groups by using some \( c \)-normal and \( c \)-supplemented subgroups.

In this paper, we remove the normal supplement condition and replace the \( c \)-supplement assumption with the \( s \)-supplement assumption for the subgroups of \( G \). We obtain the \( p \)-nilpotency of \( G \) and the related results.

All the groups in this paper are finite. Most of the notation is standard and can be found in [4] and [11].

**Definition 1.1.** A subgroup \( H \) of \( G \) is called \( s \)-supplemented in \( G \), if there exists a subgroup \( K \) of \( G \) such that \( G = HK \) and \( H \cap K \leq H_{SG} \), where \( H_{SG} \) is the largest subnormal subgroup of \( G \) contained in \( H \). In this case, \( K \) is said to be an \( s \)-supplement of \( H \) in \( G \).

Recall that a subgroup \( H \) of \( G \) is said to be \( c \)-supplemented in \( G \), if there exists a subgroup \( K \) of \( G \) such that \( G = HK \) and \( H \cap K \leq H_G \) [13]. A subgroup \( H \) is said to be \( s \)-normal in \( G \), if there exists a subnormal subgroup \( N \) of \( G \) such that \( HN = G \) and \( H \cap N \leq H_{SG} \) [14]. Hence, \( s \)-supplementation is a generalization of \( s \)-normality and \( c \)-supplementation. Moreover, we have \( s \)-supplementation cannot imply \( s \)-normality.

**Example 1.** \( A_5 = C_5A_4 \) and \( C_5 \cap A_4 = 1 \). Both \( C_5 \) and \( A_4 \) are \( c \)-supplemented in \( A_5 \) and so \( s \)-supplemented in \( A_5 \), but neither of them is \( s \)-normal in \( A_5 \), since \( A_5 \) is simple.

\( S \)-supplementation cannot imply \( c \)-supplementation.
Example 2. Let $G = Z_p \triangleleft Z_q$, where $Z_p$ and $Z_q$ are the groups of prime $p$ and $q$, respectively, $(p < q)$. Then, evidently, every subgroup $H$ of $G$ such that $H \cong Z_p$ is $s$-normal and so $s$-supplemented in $G$, but not $c$-supplemented in $G$.

2. Preliminaries

For the sake of convenience, we first list here some known results, which will be useful in the sequel.

Lemma 2.1. Let $G$ be a group. Then:

1. If $H$ is $s$-supplemented in $G$, $H \leq M \leq G$, then $H$ is $s$-supplemented in $M$.

2. Let $N \unlhd G$ and $N \leq H$. Then $H$ is $s$-supplemented in $G$, if and only if $H/ N$ is $s$-supplemented in $G/ N$.

3. Let $\pi$ be a set of primes. Let $N$ be a normal $\pi'$-subgroup and $H$ be a $\pi$-subgroup of $G$. If $H$ is $s$-supplemented in $G$, then $HN/ N$ is $s$-supplemented in $G/ N$. If furthermore $N$ normalizes $H$, then the converse also holds.

Proof. The claims in (1)-(3) are easy exercises left to the reader.

Lemma 2.2 [10, Lemma 2.7]. If $L \lhd G$ and $L$ is a $p$-subgroup, then $L \leq O_p(G)$.

Lemma 2.3. Let $\pi$ be a set of prime divisor of $|G|$. If $G \in E_\pi$, then every subnormal subgroup and every composition factor of $G$ belongs to $E_\pi$.

Proof. It is clear that every normal subgroup of $G$ belongs to $E_\pi$ when $G \in E_\pi$. For every subnormal subgroup $K$ of $G$, there exists a subnormal series of $G$

$$K = K_0 \unlhd K_1 \unlhd \cdots \unlhd K_{n-1} \unlhd K_n = G.$$
Since \( G \in E_\pi \), then \( K_{n-1} \in E_\pi \) and \( K \) belongs to \( E_\pi \) by the induction. On the other hand, it is easy to know every quotient group of \( G \) belongs to \( E_\pi \) when \( G \in E_\pi \). Similarly, every composition factor belongs to \( E_\pi \) by the induction. This completes the proof.

**Lemma 2.4.** Let \( G \) be a finite group and \( P \) be a Sylow \( p \)-subgroup of \( G \), where \( p \) is a prime divisor of \(|G|\) such that \(|G|, p - 1 = 1\). Suppose that there exists a maximal subgroup \( P_1 \) of \( P \) such that \( P_1 \) is \( s \)-supplemented in \( G \). Then \( G \) is not a non-abelian simple group and \( G \in D_p' \).

**Proof.** (1) \( G \in D_p' \).

We prove this by induction on the order of \( G \). Since \( P_1 \) is \( s \)-supplemented in \( G \), there exists a subgroup \( K \) of \( G \) such that \( P_1 K = G \) and \( P_1 \cap K \leq (P_1)_{SG} \).

If \( P_1 \cap K = 1 \), then \(|K|_p = p \). Let \( K_p \) denote a Sylow \( p \)-subgroup of \( K \). Then \( N_K(K_p) \mid C_K(K_p) \) is isomorphic to a subgroup of \( Aut(K_p) \). Hence, the order of \( N_K(K_p) \mid C_K(K_p) \) must divide \(|G|, p - 1 = 1\). Therefore, \( N_K(K_p) = C_K(K_p) \) by Burnside’s \( p \)-nilpotent theorem and hence \( K \) is \( p \)-nilpotent. It is clear that the normal \( p \)-complement \( K_p' \) is a Hall \( p' \)-subgroup of \( G \) and hence \( G \in E_{p'} \). If \( p \) is an odd prime, then \( G \) is soluble and hence \( G \in D_{p'} \). If \( p = 2 \), then [3, Main Theorem] implies that \( G \in C_2' \). By [1, P.547], if \( \pi \) is a set of odd primes and \( G \) satisfies \( E_\pi \) and \( E_{\pi'} \), then \( G \in D_{\pi'} \). Hence we have that \( G \in D_2' \).

If \( P_1 \cap K \neq 1 \) and \( K < G \), then \( P_1 \cap K = (P_1)_{SG} \cap K \ll K \). It is easy to see that \( P_1 \cap K \) is \( s \)-supplemented in \( K \). Since \(|P \cap K : P_1 \cap K| = |P : P_1| = p \), by the hypotheses, we have that \( K \in D_{p'} \). With the similar argument, we have \( G \in D_{p'} \).

Now, we may assume \( P_1 \cap K \neq 1 \) and \( K = G \), i.e., \( P_1 \ll G \). If \( p \) is an odd prime, then \( G \) is soluble since \(|G|, p - 1 = 1\) and hence \( G \in D_{p'} \). If \( p = 2 \), then there exists a subnormal series of \( G \) such that
$P_1 \unlhd M_1 \unlhd M_2 \unlhd \cdots \unlhd M_n = G$. It is easy to see that $|M_1 : P_1| = 2n_1$ or $n_2$, where $n_1$ and $n_2$ are both odd numbers. Now, we have $M_1$ is soluble. By the same argument, we obtain that $G$ is soluble. Therefore $G \in D_{p'}$.

(2) $G$ is not a non-abelian simple group.

Assume that $G$ is a non-abelian simple group. By assumption, there exists a subgroup $K$ of $G$ such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_S = 1$. In particular, $|G : K| = p^a$, $a \geq 1$. By [5, Theorem 1], we know that either $K$ is a Hall $p'$-subgroup of $G$ or $G$ is isomorphic to $PSU_4(2) \cong PSp(3)$, and $K$ is the parabolic subgroup of index 27 or $G$ is isomorphic to $A_n$ with $5 \leq n = p^r$, $r \geq 2$ and $K \cong A_{n-1}$. Clearly, $K$ is not a Hall $p'$-subgroup of $G$ since $|G : K| = |P_1 : P_1 \cap K| \leq |P_1| < |P|$. If $G \cong PSU_4(2)$, then $|G| = 2^6 \cdot 3^4 \cdot 5$ and $|K| = 2^6 \cdot 3^2 \cdot 5$. By (1) and the condition, $G \in E_{p'}$ and there exists a Hall $p'$-subgroup $G_{p'}$ of $G$ such that $|G_{p'}| = 2^6 \cdot 5$. Hence, we have $|G : G_{p'}| = 3^4$, contrary to [5, Theorem 1]. But in the last case, $|P_1| = n = p^r$, $(n! / 2) = (12 \ldots p^r) / 2$. If $r > 1$, then $p^2 | |A_{n-1}|$ and $p^2 | |P : P_1|$. Therefore, $G$ is not a non-abelian simple group.

The theorem is proved.

3. Main Results

**Theorem 3.1.** Let $G$ be a finite group and $P$ be a Sylow $p$-subgroup of $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. Suppose that every maximal subgroup of $P$ is $s$-supplemented in $G$, then $G / O_p(G)$ is $p$-nilpotent.

**Proof.** Assume that the theorem is false and choose $G$ to be a counterexample of smallest order. By Lemma 2.4, we have $G \in E_{p'}$. Furthermore, we have
(1) $O_p(G) = 1$.

If $O_p(G) = P$, then $G / O_p(G)$ is a $p'$-group and of course, it is $p$-nilpotent, a contradiction. If $1 \leq O_p(G) < P$, then $G / O_p(G)$ satisfies the hypotheses and the minimal choice of $G$ implies that $G / O_p(G) \cong G / O_p(G) / O_p(G / O_p(G))$ is $p$-nilpotent, a contradiction.

(2) For every maximal subgroup $P_1$ of $P$, the $s$-supplement of $P_1$ is $p$-nilpotent.

Let $P_1$ be a maximal subgroup of $P$. By hypotheses, $P_1$ is $s$-supplemented in $G$. So, there exists a subgroup $K_1$ of $G$ such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_G$. By Lemma 2.2, we have that $P_1 \cap K \leq (P_1)_G \leq O_p(G) = 1$. Now $|K_1|_p = p$. Let $K_{1p}$ denote the Sylow $p$-subgroup of $K_1$. Then, $N_{K_1}(K_{1p}) / C_{K_1}(K_{1p})$ is isomorphic to a subgroup of $Aut(K_{1p})$. Hence, the order of $N_{K_1}(K_{1p}) / C_{K_1}(K_{1p})$ must divide $(|G|, p - 1) = 1$. Therefore $N_{K_1}(K_{1p}) = C_{K_1}(K_{1p})$. Burnside’s $p$-nilpotent theorem [11, 10.1.8] implies that $K_1$ is $p$-nilpotent.

(3) $G$ is $p$-nilpotent.

Let $P_1$ be a maximal subgroup of $P$. By (1) and (2), there exists a $p$-nilpotent subgroup $K_1$ of $G$ such that $G = P_1K_1$. Let $K_1 = K_1K_1p'$ and $N = N_G(K_1p')$. Clearly, $K_1 \leq N$ and $G = PN$. If $P \leq N$, then $N = G$, a contradiction. So, we may assume that $P \cap N < P$. There exists a maximal subgroup $P_2$ of $P$ such that $P \cap N \leq P_2$. By hypotheses, $P_2$ is $s$-supplemented in $G$. (2) indicates that the supplement $K_2$ of $P_2$ is $p$-nilpotent. We denote $K_2 = K_2K_2p'$. Now both $K_1p'$ and $K_2p'$ are Hall $p'$-subgroup of $G$. Since $(|G|, p - 1) = 1$, by Lemma 2.4, we have $G \in Dp'$, these two subgroups are conjugate in $G$. Say $K_1p' = (K_2p')^g$.

Since $G = P_2K_2$ and $K_2p' \leq K_2$, we may choose $g \in P_2$. $K_2^g$ normalizes
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Based on the discussion as above and [2], $G/O_p(G)$ is $p$-nilpotent.

**Lemma 3.2** [9, Lemma 2.4]. Let $G$ be a finite group and $p$ be a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. Assume that the order of $G$ is not divisible by $p^3$. Then $G$ is $p$-nilpotent.

**Theorem 3.3.** Let $G$ be a finite group and $p$ be a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. Assume that every second maximal subgroup of the Sylow $p$-subgroup of $P$ is $s$-supplemented in $G$, then $G/O_p(G)$ is soluble $p$-nilpotent.

**Proof.** Assume that the claim is false and choose $G$ to be a counterexample of minimal order. Furthermore, we have

1. $O_p(G) = 1$.

   If $O_p(G) = P$, then $G/O_p(G)$ is a $p'$-group and of course, it is $p$-nilpotent, a contradiction. If $O_p(G) = P_1$, where $P_1$ is the maximal subgroup of $P$, then $G/O_p(G)$ is $p$-nilpotent since $(|G|, p - 1) = 1$ and $|G/O_p(G)|_p = p$, a contradiction. If $O_p(G) = P_2$, where $P_2$ is the second maximal subgroup of $P$, then $p^3 | |G/O_p(G)|$. Hence, $G/O_p(G)$ is $p$-nilpotent by Lemma 3.2. If $1 < O_p(G) < P_2$, then $G/O_p(G)$ satisfies the hypotheses and the minimal choice of $G$ implies that $G/O_p(G) \cong G/O_p(G)/O_p(G/O_p(G))$ is $p$-nilpotent, a contradiction.

2. $|G|$ is divisible by $p^3$.

   If $p^3 | |G|$, then $G$ is $p$-nilpotent by Lemma 3.2, a contradiction.
(3) For every second maximal subgroup $P_1$ of a Sylow subgroup $P$ of $G$, the $s$-supplement of $P_1$ is $p$-nilpotent.

Let $P$ be a Sylow $p$-subgroup of $G$ and $P_1$ be a second maximal subgroup of $P$. By hypotheses, $P_1$ is $s$-supplemented in $G$. So, there exists a subgroup $K_1$ of $G$ such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_{SG}$. By Lemma 2.2, we have $P_1 \cap K_1 \leq (P_1)_{SG} \leq O_p(G) = 1$. Now $|K_1|_p = p^2$.

By hypotheses and Lemma 3.2, we have $K_1$ is $p$-nilpotent.

(4) $G$ is $p$-nilpotent.

Let $N = N_G(K_{1_{p'}})$ and $K_1 = K_{1_{p'}}K_{1_{p'}}$. By (3), $K_1 \leq N$. So, we have $G = P_1K_1 = P_1N$. If $N = G$, then $G$ is $p$-nilpotent, a contradiction. Let $P_1 \leq \overline{P}_1 \leq P$, where $\overline{P}_1$ is a maximal subgroup of Sylow subgroup $P$ of $G$. Hence, $G = P_1K_1 = \overline{P}_1K_1 = \overline{P}_1N$. If $\overline{P}_1 \leq N$, then $G$ is $p$-nilpotent, a contradiction. So, we may assume $\overline{P}_1 \cap N < \overline{P}_1$. We may choose a maximal subgroup $P_2$ of $\overline{P}_1$ such that $\overline{P}_1 \cap N \leq P_2$. It is clear that $P_2$ is the second maximal subgroup of $P$. By (3), $P_2$ is $s$-supplemented in $G$ and the supplement $K_2$ of $P_2$ is $p$-nilpotent. We denote $K_2 = K_{2_{p'}}K_{2_{p'}}$.

Since $(|G|, p^2 - 1) = 1$, [3, Main Theorem] or the odd order theorem [2] implies that $G \in C_{p'}$. Now both $K_{1_{p'}}$ and $K_{2_{p'}}$ are Hall $p'$-subgroup of $G$, these two subgroups are conjugate in $G$. Let $K_{1_{p'}} = (K_{2_{p'}})^g$. Since $G = P_2K_2$ and $K_2 \leq N_G(K_{2_{p'}})$, we may choose $g \in P_2$. $K_{2_{p'}}^g$ normalizes $K_{2_{p'}}^g = K_{1_{p'}}^g$ and hence $K_{2_{p'}}^g \leq N$. Now $G = (P_2K_2)^g = P_2N$. Therefore $\overline{P}_1 = \overline{P}_1 \cap P_2N = P_2(\overline{P}_1 \cap N) = P_2$, contrary to the condition.

The final contradiction completes our proof.
**Theorem 3.4.** Let $G$ be a finite group. Then $G$ is soluble, if and only if every Sylow subgroup of $G$ is $s$-supplemented in $G$.

**Proof.** If $G$ is soluble, then by [6, Main Theorem], every Sylow subgroup of $G$ is complemented in $G$. It is clear that every Sylow subgroup of $G$ is $s$-supplemented in $G$.

Conversely, assume that every Sylow subgroup $P$ of $G$ is $s$-supplemented in $G$. By [6, Main Theorem], we only need to prove that $P$ is complemented in $G$. Let $K$ be an $s$-supplement of $P$ in $G$. Then $G = PK$ and $P \cap K \leq P_{SG}$.

If $P \cap K = 1$, then $P$ is complemented in $G$.

If $P \cap K \neq 1$, then $P \cap K = P_{SG} \cap K \lhd K$. Note that $|G|_p = |P| / |P_{SG} \cap K|$, hence $|K|_p = |P_{SG} \cap K|$ and $P \cap K = P_{SG} \cap K \leq K$. By the Schur-Zassenhaus theorem [11, Theorem 9.1.10], we have that $K = (P \cap K)K_{p'}$, where $K_{p'}$ is the Hall $p'$-subgroup of $K$.

Now, $G = PK = P(P \cap K)K_{p'} = PK_{p'}$ and $P \cap K_{p'} = 1$. Therefore, $P$ is complemented in $G$. The theorem is proved.

**References**


